UFIFT-HEP-00-18
hep-th/0007192

A Note on Non-Locality and Ostrogradski's Construction

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July 2000

Abstract

We consider the Hamiltonian treatment of non-local theories and Ostrogradski's formalism. This has recently also been discussed by Woodard (hep-th/0006207) and by Gomis, Kamimura and Llosa (hep-th/0006235). In our approach we recast the second class constraints into first class constraints and invoke the boundary Poisson bracket.

PACS number(s): 02.70.Pt, 03.50.-z, 04.20.Fy, 11.10.Ef, 11.10.Lm, 45.20.Jj.
Keywords: Non-locality, Poisson Bracket, Boundary Term, Ostrogradski, Functional Derivative.

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1 Motivations

Woodard's construction [1] has boundaries in the non-local integration range at $x = 0$ and $x = \Delta t$. If caution is not applied, a breakdown of the Jacobi identity and a loss of differentiability of functionals may lead to inconsistencies. As mentioned in [1], additional boundary conditions must be imposed. Intuitively, the boundaries of the non-local integration range should not have any physical significance. We shall devise a slightly modified construction that avoids boundaries and hence steers clear of such hazards.

In addition, we are interested in the close relationship between Ostrogradski's derivatives and the higher Euler-Lagrange derivatives used among other things to construct boundary Poisson brackets, cf. [7, 9, 8, 10].

Compared with earlier work, we do not have second class constraints. We obtain a clear separation of the equations of motion from the constraints, which is preferable at the conceptual level.

We shall have nothing to say about the soundness of non-local theories and higher derivative theories in general. Instead we refer to the ongoing discussion in the Literature [1, 3].

2 Lagrangian Variables

For simplicity, we shall consider a $0+1$ dimensional systems with one dynamical variable $q(t)$. The construction can be straightforwardly extended to include more variables and to field theories. We shall assume that the time coordinate t has no temporal boundaries, *i.e.* $t \in I\!R$ can take any real value.

We are free to think [4] of the dynamical variable $q(t)$ as a $1+1$ dimensional field $Q(x,t)$ that satisfies a chirality condition

$$\frac{d}{dt}Q(x,t) \;=\; \partial_x Q(x,t) \; . \tag{2.1}$$

Explicitly, the one-to-one correspondence between $q(t)$ and the left-mover $Q(x,t)$ is given by $Q(x,t) = q(x+t)$. Keep in mind for later that eq. (2.1) implies

$$\left(\frac{d}{dt}\right)^n Q(x,t) \;=\; (\partial_x)^n Q(x,t) \; , \quad n \in I\!N_0 \; . \tag{2.2}$$

3 Lagrangian

By *non-locality* we mean that the Lagrangian $L[q](t)$ depends on the dynamical variable q at other times than t. To deal with this in a systematic manner we shall assume that the Lagrangian can be written as a d-dimensional integral

$$L[Q](t) \;=\; \int_{-\infty}^{\infty} dx^1 \cdots \int_{-\infty}^{\infty} dx^d \; \mathcal{L}(x^1, \ldots, x^d, t) \tag{3.1}$$

over a density function $\mathcal{L}$. To be precise, besides the explicit dependence of $x^1, \ldots, x^d$ and t, the density function $\mathcal{L}(x^1, \ldots, x^d, t)$ is assumed to be a function of a *finite* number of the following variables:

$$(\partial_x)^k Q(x^i, t) \ , \quad i = 1, \ldots, d, \ , \quad k \in \mathbb{N}_0 \ . \tag{3.2}$$

The replacement of $q(t)$ with $Q(x, t)$ has several advantages:

First of all, $L[Q](t)$ does only depend on $Q(x, t)$'s of the very same t. The non-locality is encoded in the new variable x. Negative and positive values of x correspond to interactions with the past and the future, respectively. (We should stress that x has nothing to do with space; we have merely named the new variable x because the formulas fit the framework of field theory.)

Secondly, in the Q-formulation we have removed all derivatives wrt. t which appeared in the original Lagrangian $L[q](t)$ by using the chirality condition (2.2). This will prepare us for a very smooth transition into its Hamiltonian counterpart, *i.e.* the Lagrangian does not depend on the velocities, on the accelerations, etc. and hence on the momenta. (Note that in the Hamiltonian formulation our starting point will be $L[Q](t)$ *without* assuming chirality of $Q(x, t)$. We shall later see that in the Hamiltonian formulation the chirality condition (2.1) becomes the equation of motion for $Q(x, t)$.)

4 Local Field Theory

A functional (3.1) with the assumption that its density $\mathcal{L}(x^1, \ldots, x^d, t)$ depends on a *finite* number of variables from the list (3.2) is commonly known as the very definition of a *local* functional. We may say that the original *non-local* theory has become a *local field* theory.

The case of *discrete* non-locality has been studied extensively in the Literature [1, 3]. We define it as the case where there exists a discrete family of curves $\{t \mapsto x_i(t)\}_{i \in I}$ so that $L[Q](t)$ only depends on a *finite* number of the following variable: t, $x_i(t)$ and

$$(\partial_x)^k Q(x_i(t), t) \ , \quad i \in I \quad , \quad k \in \mathbb{N}_0 \ . \tag{4.1}$$

For technical reasons we shall assume the function $\mathcal{L}(x^1, \ldots, x^d, t)$ is C^∞-differentiable. This smoothness assumption unfortunately rules out the case of discrete non-locality. However, we shall investigate the discrete case in Section 13.

5 Compact Support

We assume that the density function $\mathcal{L}$ (and other physically meaningful objects) has a *compact* support in the x-directions. As we shall see this assumption has radical implications for the theory.

6 Momenta

In a non-local Lagrangian theory the usual definition of momenta as the derivatives of the Lagrangian wrt. the velocities is not useful. Instead, we shall seek a new and better definition. We

take as our initial guess the partial differential equation

$$\frac{d}{dt}P(x,t) \;=\; \partial_x P(x,t) + \frac{\delta L[Q](t)}{\delta Q^{(0)}(x,t)} \;, \tag{6.1}$$

subjected to the following boundary condition:

$$P(x,t) \text{ has compact support in the } x\text{−direction.} \tag{6.2}$$

The (0) appearing on the symbol for the functional derivatives denotes that we use the algebraic Euler-Lagrange definition rather than a variational definition of the functional derivatives, cf. [9]. We shall see below that our initial guess (6.1) needs off-shell modifications.

Both the partial differential equation (6.1) and the boundary condition (6.2) appear naturally in the Hamiltonian treatment to be given below. Purely from a Lagrangian perspective, the boundary condition (6.2) arises as a natural consequence of requiring the density function $\mathcal{L}$ itself to have compact support.

In general the above boundary value problem (6.1) and (6.2) may not have continuous solutions. Let us allow for a potential discontinuity along a curve $x = x_0(t)$. The unique solution is then [4]

$$\begin{aligned} P(x,t) \;&=\; \int_{-\infty}^{\infty}\!dx' \int_{-\infty}^{\infty}\!dt'\; \delta_{I\!R}(x'+t'-x-t)\,\left[\theta(x'-x)\;\theta(x-x_0(t)) - \theta(x_0(t)-x)\;\theta(x-x')\right] \frac{\delta L[Q](t')}{\delta Q^{(0)}(x',t')} \\ &=\; \int_{-\infty}^{\infty}\!du\;\left[\theta(x-x_0(t))\;\theta(u) - \theta(x_0(t)-x)\;\theta(-u)\right] \frac{\delta L[Q](t-u)}{\delta Q^{(0)}(x+u,t-u)} \;. \end{aligned} \tag{6.3}$$

In a Lagrangian treatment we promote this formula to be the very definition of momenta (see [1, formula (20)] and [2, formula (6)]). Needless to say, formula (6.3) may have other discontinuities if $\delta L[Q](t)/\delta Q^{(0)}(x,t)$ is singular, cf. Section 13.

7 Lagrangian Equation of Motion

Let $S[q] = \int_{-\infty}^{\infty} dt\; L[q](t)$ denote the action. Recall that we cannot vary Q freely because of the chirality condition eq. (2.1). Therefore, the Lagrangian Equation of Motion for Q does not provide us with the relevant physical information. Instead, the pertinent equation of motion is given by the Lagrangian Equation of Motion for q:

$$\begin{aligned} 0 \;&=\; \frac{\delta S[q]}{\delta q^{(0)}(x+t)} \;\stackrel{(2.1)}{=}\; \int_{-\infty}^{\infty}\!du\; \frac{\delta L[Q](t-u)}{\delta Q^{(0)}(x+u,t-u)} \\ &\equiv\; \int_{-\infty}^{\infty}\!dx' \int_{-\infty}^{\infty}\!dt'\; \delta_{I\!R}(x'+t'-x-t)\; \frac{\delta L[Q](t')}{\delta Q^{(0)}(x',t')} \;. \end{aligned} \tag{7.1}$$

In a Q-formulation, with the chirality condition eq. (2.1) not imposed manifestly, the Equation of Motion for q does strictly speaking not make sense (at least not a priori). However, even in that case, it is natural to call the last two expressions in eq. (7.1) the Lagrangian Equation of Motion for q.

Note that eq. (7.1) is precisely the condition that the momentum formula (6.3) does not have an extra discontinuity at $x = x_0(t)$:

$$\lim_{x\to x_0(t)^-} P(x,t) \;=\; \lim_{x\to x_0(t)^+} P(x,t) \;. \tag{7.2}$$

From a Hamiltonian point of view, where the momentum is a fundamental rather than a derived quantity, this is of course trivially guaranteed by restricting ourselves to continuous fields. (In fact we shall only allow C^{∞}-fields.)

Closely related to this fact is the following: If we include singular terms at the kink curve, the above momentum formula (6.3) satisfies the following partial differential equation:

$$\frac{d}{dt}P(x,t) \;=\; \partial_x P(x,t) + \frac{\delta L[Q](t)}{\delta Q^{(0)}(x,t)} + (\dot{x}_0(t)-1)\;\delta_{I\!R}(x-x_0(t))\;\frac{\delta S[q]}{\delta q^{(0)}(x+t)}\;. \tag{7.3}$$

This differs from the original eq. (6.1) with a delta function contribution along the curve that is proportional to the Lagrangian Equation of Motion. The extra term also vanishes along equal-time curves $x_0(t)+t=\text{constant}$.

8 Gauge Symmetry

We will now discuss the Hamiltonian formulation. As mention earlier, our starting point is the Lagrangian $L[Q](t)$ *without* assuming chirality of $Q(x,t)$. The newly gained freedom of the $Q(x,t)$-fields introduces a gauge symmetry for the the Lagrangian $L[Q](t)$ in the following way; for a given time t, let (for the time being) Σ_t denote the support

$$\mathrm{supp}\left(x\mapsto\frac{\delta L[Q](t)}{\delta Q^{(0)}(x,t)}\right) \;\equiv\; \overline{\left\{x\in I\!R\;\middle|\;\frac{\delta L[Q](t)}{\delta Q^{(0)}(x,t)}\neq 0\right\}} \tag{8.1}$$

of the Euler-Lagrange function

$$x \;\;\mapsto\;\; \frac{\delta L[Q](t)}{\delta Q^{(0)}(x,t)}\;\;. \tag{8.2}$$

It follows from previously made assumptions that Σ_t is compact. The Lagrangian $L[Q](t)$ will be invariant under all transformations $\delta Q(x,t)$ that leaves Σ_t invariant:

$$\forall x\in\Sigma_t\;:\;\delta Q(x,t)\;=\;0\;. \tag{8.3}$$

The value of the field $Q(x,t)$ for $x\notin\Sigma_t$ has no physical content. It represents a gauge degree of freedom for the system.

9 Boundary Poisson Bracket

We take the boundary Poisson bracket [7, 9, 8, 10] for local functionals $F(t)$ and $G(t)$ to be given by the following ultra-local ansatz [10]:

$$\{F(t),G(t)\} \;=\; \sum_{k,\ell=0}^{\infty} c_{k,\ell}\int_{-\infty}^{\infty}\!dx\;(\partial_x)^{k+\ell}\left[\frac{\delta F(t)}{\delta Q^{(k)}(x,t)}\;\frac{\delta G(t)}{\delta P^{(\ell)}(x,t)}\right]\;-\;(F\leftrightarrow G)\;. \tag{9.1}$$

Here $\delta/\delta Q^{(k)}(x,t)$ are the higher Euler-Lagrange derivatives, cf. [9], and the coefficients $c_{k,\ell}$ are constants. They are normalized such that $c_{0,0}=1$ and such that the Jacobi identity is satisfied. In particular, one may show that

$$c_{k,0} \;=\; 1 \;=\; c_{0,\ell}\;. \tag{9.2}$$

We can extract the usual canonical equal-t relation from (9.1):

$$\{\;Q(x,t),\;P(x',t)\;\} \;=\; \delta_{I\!R}(x-x')\;. \tag{9.3}$$

10 First Class Constraints

The gauge symmetry is generated by the following first class contraints

$$\forall x \notin \Sigma_t \ : \ P(x,t) \ \approx \ 0 \ , \tag{10.1}$$

which is the Hamiltonian version of the boundary condition (6.2). (The wavy double line $\approx$ is a notation first introduced by Dirac to denote equality modulo first class constraints, so-called *weak* equality.) To check in detail that the first class constraint (10.1) generates the gauge transformations (8.3), consider the smeared first class constraint

$$T[\xi](t) \ \equiv \ \int_{-\infty}^{\infty} dx \ \xi(x,t) \ P(x,t) \ , \tag{10.2}$$

where $\xi(x,t)$ is a test function that vanishes on Σ_t :

$$\forall x \in \Sigma_t \ : \ \xi(x,t) \ = \ 0 \ . \tag{10.3}$$

Here $\xi(x,t)$ does not depend on the dynamical variables $P(x,t)$ and $Q(x,t)$. Let $\delta Q = \xi$ be a infinitesimal gauge transformation. The gauge variation of the local functional $F[Q,P](t)$ is given by

$$\begin{aligned} \delta_\xi F[Q,P](t) \ &\equiv \ F[Q{+}\xi,P](t) - F[Q,P](t) \\ &= \ \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} dx \ (\partial_x)^k \left[\frac{\delta F[Q,P](t)}{\delta Q^{(k)}(x,t)} \ \xi(x,t) \right] \\ &= \ \{ \ F[Q,P](t), \ T[\xi](t) \ \} \ . \end{aligned} \tag{10.4}$$

In the last equality, we used (9.2).

11 Hamiltonian

The bare action S and the bare Hamiltonian $H(t)$ is given by

$$\begin{aligned} S \ &= \ \int_{-\infty}^{\infty} dt \left[\int_{-\infty}^{\infty} dx \ P(x,t) \ \dot{Q}(x,t) \ - \ H(t) \right] \\ H(t) \ &= \ \int_{-\infty}^{\infty} dx \ P(x,t) \ \partial_x Q(x,t) \ - \ L[Q](t) \ . \end{aligned} \tag{11.1}$$

The Hamiltonian Equation of Motion

$$\dot{F}(t) \ \approx \ \{ \ F(t), \ H(t) \ \} \tag{11.2}$$

for a local functional $F(t)$ becomes

$$\begin{aligned} &\sum_{k=0}^{\infty} \int_{-\infty}^{\infty} dx \ (\partial_x)^k \left[\frac{\delta F(t)}{\delta Q^{(k)}(x,t)} \ \dot{Q}(x,t) + \frac{\delta F(t)}{\delta P^{(k)}(x,t)} \ \dot{P}(x,t) \right] \\ &\approx \ \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} dx \ (\partial_x)^k \left[\frac{\delta F(t)}{\delta Q^{(k)}(x,t)} \ \partial_x Q(x,t) + \frac{\delta F(t)}{\delta P^{(k)}(x,t)} \ \left(\partial_x P(x,t) + \frac{\delta L[Q](t)}{\delta Q^{(0)}(x,t)} \right) \right] \end{aligned}$$

$$
-\sum_{k=0}^{\infty} c_{1,k} \int_{-\infty}^{\infty} dx\ (\partial_x)^{k+1} \left[\frac{\delta F(t)}{\delta P^{(k)}(x,t)}\ P(x,t) \right] \ , \tag{11.3}
$$

where we have used (9.2) and the fact that $\mathcal{L}$ has compact support. If we furthermore apply the first class constraints (10.1), we can get rid of the last term on the right hand side. The remaining equation is clearly equivalent to the chirality condition eq. (2.1) for $Q(x,t)$ and eq. (6.1) for $P(x,t)$.

One may indeed check that the first class constraint (10.1) is preserved under the Hamiltonian flow, as it should be:

$$
\begin{aligned}
\{\ T[\xi](t),\ H(t)\ \} &= \int_{-\infty}^{\infty} dx\ \xi(x,t) \left(\partial_x P(x,t) + \frac{\delta L[Q](t)}{\delta Q^{(0)}(x,t)} \right)\ \approx\ 0\ , \\
\delta_\xi S &= \int_{-\infty}^{\infty} dt \left[-\int_{-\infty}^{\infty} dx\ \xi(x,t)\dot{P}(x,t) + \{\ T[\xi](t),\ H(t)\ \} \right]\ \approx\ 0\ .
\end{aligned} \tag{11.4}
$$

Finally, an important remark: We could generalize the previous discussion by letting Σ_t be *any* compact set which contains the support (8.1), thereby deliberately choosing a smaller gauge symmetry. In fact, a transformation $\delta Q(x,t)$ which leaves the support (8.1) – but not the larger set Σ_t – invariant, would no longer be a symmetry for the action S nor the Hamiltonian $H(t)$. The first class constraint (10.1) is creating its own justification! This is because the original Lagrangian theory – build out of rigid left-movers – does not possess any gauge symmetry at all. We see that the theory changes with the choice of Σ_t. The smaller we choose Σ_t, the more the system is prohibited in sending left-moving momenta between different connected components of the support (8.1). On the other hand, we do not want to completely eliminate the gauge symmetry by choosing Σ_t as large as possible, *i.e.* $\Sigma_t = I\!R$. That would complicate matters by activating the extra boundary terms appearing in the equations of motion (11.3). In conclusion, to ensure that our Hamiltonian system corresponds to the original Lagrangian theory, we let Σ_t *be a compact set bigger than the convex hull* of the support (8.1).

12 Ostrogradski's Framework

Let us recall how the non-local formulation translates into Ostrogradski's formulation [6] of infinite order. For other treatments, see [5, Appendix A], [4, Section VI A] and [1, Section 5]. We assume for simplicity that the discontinuity curve $x_0(t) = x_0$ is constant. Ostrogradski's coordinates $Q^{(n)}(t)$, $n \in I\!N_0$, are defined as

$$
Q^{(n)}(t)\ =\ (\partial_x)^n Q(x,t)\big|_{x=x_0}\ . \tag{12.1}
$$

The inverse relation is given by the Taylor expansion around $x = x_0$:

$$
Q(x,t)\ =\ \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} Q^{(n)}(t)\ . \tag{12.2}
$$

The chirality condition eq. (2.1) translates into

$$
\dot{Q}^{(n)}(t)\ =\ Q^{(n+1)}(t)\ . \tag{12.3}
$$

To ensure that the boundary Poisson bracket eq. (9.1) corresponds to the discrete analogue given by

$$
\{Q^{(n)}(t), P_{(m)}(t)\}\ =\ \delta^n_m\ , \tag{12.4}
$$

we define Ostrogradski's momenta $P_{(n)}(t)$ as

$$P_{(n)}(t) \;=\; \int_{-\infty}^{\infty}\! dx \; \frac{(x-x_0)^n}{n!} \; P(x,t) \; . \tag{12.5}$$

The integral is well-defined because the momenta $P(x,t)$ have compact support, cf. eq. (6.2). The inverse relation reads

$$P(x,t) \;=\; \sum_{n=0}^{\infty} P_{(n)}(t) \; (-\partial_x)^n \delta_{I\!R}(x-x_0) \; . \tag{12.6}$$

Alternatively, the formulas for the momenta follows from the Schrödinger representation

$$\frac{\delta}{\delta Q^{(0)}(x,t)} \;=\; \sum_{n=0}^{\infty} (-\partial_x)^n \delta_{I\!R}(x-x_0) \; \frac{\partial}{\partial Q^{(n)}(t)} \; , \tag{12.7}$$

and equivalently

$$\frac{\partial}{\partial Q^{(n)}(t)} \;=\; \int_{-\infty}^{\infty}\! dx \; \frac{(x-x_0)^n}{n!} \; \frac{\delta}{\delta Q^{(0)}(x,t)} \; . \tag{12.8}$$

The equations (6.3) and (6.1) translate into

$$P_{(n)}(t) \;=\; \sum_{m=n}^{\infty} (-\partial_t)^{m-n} \frac{\partial L[Q](t)}{\partial Q^{(m+1)}(t)} \tag{12.9}$$

and

$$\left\{ \begin{array}{rcl} \dot{P}_{(n)}(t) + P_{(n-1)}(t) &=& \frac{\partial L[Q](t)}{\partial Q^{(n)}(t)} \; , \quad n \in I\!N \; , \\ \\ \dot{P}_{(0)}(t) &=& \frac{\partial L[Q](t)}{\partial Q^{(0)}(t)} \; , \end{array} \right. \tag{12.10}$$

respectively. The very last equation is the Lagrangian Equation of Motion. The Hamiltonian (11.1) translates into

$$H(t) \;=\; \sum_{n=0}^{\infty} P_{(n)}(t) \; Q^{(n+1)}(t) \;-\; L[Q](t) \; . \tag{12.11}$$

The Hamiltonian Equations of Motion are (12.3) and (12.10).

13 Discrete Case

Finally, let us consider the discrete case, cf. Section 4, with constant curves $x_i(t) = x_i$. The Euler-Lagrange equation for Q reads

$$\begin{aligned} \frac{\delta L[Q](t)}{\delta Q^{(0)}(x,t)} &= \sum_{i\in I} \sum_{k=0}^{\infty} \frac{\partial L[Q](t)}{\partial Q^{(k)}(x_i,t)} \; (-\partial_x)^k \delta_{I\!R}(x-x_i) \\ &= \sum_{i\in I} \sum_{k=0}^{\infty} E_{(k)}(x_i, t+x-x_i) \; (-\partial_x)^k \delta_{I\!R}(x-x_i) \; . \end{aligned} \tag{13.1}$$

Here we have introduced the higher Euler-Lagrange derivatives, cf. [9]:

$$E_{(k)}(x,t) \;=\; \sum_{m=k}^{\infty} \left(\begin{array}{c} m \\ k \end{array} \right) (-\partial_t)^{m-k} \frac{\partial L[Q](t)}{\partial Q^{(m)}(x,t)} \quad , \qquad x = x_i \quad , \qquad k \in Z\!\!\!Z \quad . \tag{13.2}$$

An alternative basis is provided by the Ostrogradski derivatives:

$$O_{(k)}(x,t) \;=\; \sum_{m=k}^{\infty}(-\partial_t)^{m-k}\frac{\partial L[Q](t)}{\partial Q^{(m)}(x,t)} \;, \qquad x=x_i \;, \qquad k\in\mathbb{Z} \;. \tag{13.3}$$

The partial derivatives can be recovered via the inverse relation

$$\frac{\partial L[Q](t)}{\partial Q^{(k)}(x,t)} \;=\; O_{(k)}(x,t)+\partial_t O_{(k+1)}(x,t) \;, \qquad x=x_i \;, \qquad k\in\mathbb{Z} \;. \tag{13.4}$$

We assume that these are regular C^{∞}-functions. Note the peculiar fact, that although the Lagrangian $L[Q](t)$ contains no temporal derivatives ∂_t – only spatial derivatives – it is the temporal derivatives ∂_t that is used in the construction of the above functions. Of course, this is the same on-shell, *i.e.* when (2.1) holds.

The momentum formula (6.3) becomes:

$$\begin{aligned}
P(x,t) \;&=\; \theta(x-x_0(t))\sum_{k=0}^{\infty}(-\partial_x)^k\sum_{i\in I}\left[\theta(x_i-x)\;\frac{\partial L[Q](t+x-x_i)}{\partial Q^{(k)}(x_i,t+x-x_i)}\right] \\
&\quad -\;\theta(x_0(t)-x)\sum_{k=0}^{\infty}(-\partial_x)^k\sum_{i\in I}\left[\theta(x-x_i)\;\frac{\partial L[Q](t+x-x_i)}{\partial Q^{(k)}(x_i,t+x-x_i)}\right] \\
&=\; \sum_{i\in I}\sum_{k=0}^{\infty}E_{(k)}(x_i,t+x-x_i)\;\left[\theta(x-x_0(t))\;(-\partial_x)^k\theta(x_i-x)-\theta(x_0(t)-x)\;(-\partial_x)^k\theta(x-x_i)\right] \\
&=\; \sum_{i\in I}\sum_{k=1}^{\infty}E_{(k)}(x_i,t+x-x_i)\;(-\partial_x)^{k-1}\delta_{\mathbb{R}}(x-x_i) \\
&\quad +\sum_{i\in I}E_{(0)}(x_i,t+x-x_i)\;\left[\theta(x_i-x)\;\theta(x-x_0(t))-\theta(x_0(t)-x)\;\theta(x-x_i)\right] \\
&=\; \sum_{i\in I}\sum_{k=1}^{\infty}O_{(k)}(x_i,t)\;(-\partial_x)^{k-1}\delta_{\mathbb{R}}(x-x_i) \\
&\quad +\sum_{i\in I}O_{(0)}(x_i,t+x-x_i)\;\left[\theta(x_i-x)\;\theta(x-x_0(t))-\theta(x_0(t)-x)\;\theta(x-x_i)\right] \;.
\end{aligned} \tag{13.5}$$

We learn that the typical momenta for a discrete system will be distributional in nature with non-smooth behavior at the discrete points $x=x_i$. Note that the support of the momenta $x\mapsto P(x,t)$ is inside the convex hull of the support (8.1) ($=\{x_i|i\in I\}$), if $x_0(t)$ is.

The Lagrangian Equation of Motion for q, cf. eq. (7.1), reads:

$$0 \;=\; \frac{\delta S[q]}{\delta q^{(0)}(x+t)} \;\stackrel{(2.1)}{=}\; \sum_{i\in I}E_{(0)}(x_i,t+x-x_i) \;. \tag{13.6}$$

14 Discrete Local Formulation

We continue to study the discrete case. We have already given a local field formulation above. But after all, it is awkward to use field theory to quantize a discrete system! It is natural to ask if one can get a local formulation *without* introducing fields? Ostrogradski's formal approach, cf. Section 12, is cumbersome in practice if the index set I contains more than one element.

If we wish to address the original problem – and hence choosing Σ_t to be the convex hull of $\{x_i|i\in I\}$ – there seems to be no easy/useful/natural local discrete formulation in general.

However, if we let Σ_t be the smallest set possible, *i.e.* $\{x_i|i\in I\}$, we can do better. The first class constraints (10.1) along with the equations of motion (6.1) would then exclude the last term involving the Heaviside step function θ in the last two expressions of (13.5). It implies an Euler-Lagrange equation for each individual $i\in I$:

$$O_{(0)}(x_i,t) \;\equiv\; E_{(0)}(x_i,t) \;=\; 0\;. \tag{14.1}$$

This corresponds to a theory *without* the chirality condition (2.1). We can fit this inside a generalized Ostrogradski framework with new coordinates $Q^{i(n)}(t)$, $i\in I$, $n\in I\!N_0$, defined as

$$Q^{i(n)}(t) \;=\; (\partial_x)^n Q(x,t)|_{x=x_i}\;. \tag{14.2}$$

and momenta $P_{i(n)}(t)$, $i\in I$, $n\in I\!N_0$,

$$P_{i(n)}(t) \;=\; \lim_{\epsilon\to 0^+}\int_{x_i-\epsilon}^{x_i+\epsilon} dx\;\frac{(x-x_i)^n}{n!}\;P(x,t)\;, \tag{14.3}$$

canonical Poisson bracket

$$\{Q^{i(n)}(t),P_{j(m)}(t)\} \;=\; \delta^n_m\;\delta^i_j\;, \tag{14.4}$$

and Hamiltonian

$$H(t) \;=\; \sum_{i\in I}\sum_{n=0}^{\infty} P_{i(n)}(t)\;Q^{i(n+1)}(t)\;-\;L[Q](t)\;. \tag{14.5}$$

More relations can be read off from the Section 12 with the obvious minor modifications. In this way we have obtained an unconstrained discrete Hamiltonian system with slightly generalized Ostrogradski coordinates. For instance, the classical momentum formula (13.5) turns into

$$P_{i(n)}(t) \;=\; O_{(n+1)}(x_i,t)\;,\qquad i\in I\;,\qquad n\in I\!N_0\;. \tag{14.6}$$

Acknowledgements. We would like to thank R.P. Woodard and B.D. Baker for comments. The research is supported by DoE grant no. DE-FG02-97ER-41029.

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